Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Summer Examination-2016

Subject Name : Operator Theory

Subject Code : 5SC04OTE1 **Branch: M.Sc.(Mathematics)**

Semester : 4 **Date : 10/05/2016** **Time : 02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions.** **(07)**

a. Define : Resolution of Identity. **(02)**
b. Is the following statement true or false? If P is self adjoint then P is normal. **(01)**
c. If $S,T \in B(H)$ and $< Sx, x >=< Tx, x >$ then show that $S = T$ for every$x \in H$. **(01)**
d. Write definition of Unitary operator. **(01)**
e. An operator $T \in B(H)$ and $\|Tx\| = \|T^*x\|$ then what can we say about T? **(01)**
f. Define: $R_{ess}(f)$. **(01)**

Q-2 **Attempt all questions** **(14)**

a) Assume that $M, N, T \in B(H)$, M and N are normal and $MT = TN$ then show that $M^*T = TN^*$ **(07)**
b) Prove that : Every non empty closed convex set $E \subset H$ contains a unique x of minimal norm. **(07)**

OR

Q-2 **Attempt all questions** **(14)**

a) If T is normal , show that T is invertible if and only if there exists $\delta > 0$ such that $\|Tx\| \geq \delta\|x\|$ .Also show that If $Tx = \alpha x$, then $T^*x = \bar{\alpha}x$. **(07)**
b) Let $U \in B(H)$ be unitary operator and $\lambda \in \sigma(U)$ then show that $|\lambda| = 1$. **(07)**

Q-3 **Attempt all questions** **(14)**

a) If E is resolution of identity and if $x \in H$, show that $\omega \longmapsto E(\omega)x$ is a countably additive H-valued measure on $\mathcal{m}$. **(07)**
b) Show that $||| f + N ||| = \|f\|_\infty$, where $f \in B_b(\Omega)$. **(07)**

OR

Q-3 **Attempt all Questions** **(14)**

a) State and prove Spectral Theorem.

b) Prove that : Let $T \in B(H)$ be normal , there exists a unique resolution of identity E on $B(\sigma(T))$ such that $T = \int_{\sigma(T)} \lambda \, dE(\lambda) = \int_{\sigma(T)} i_{\sigma(T)}(\lambda) dE(\lambda)$

SECTION – II

Q-4 **Attempt the Following questions.** **(07)**

a. Following statement is true or false? If $T \in B(H)$ is normal and self adjoint then $\sigma(T)$ lies in the real axis. **(01)**

b. Write definition of polar decomposition of T. **(01)**

c. Define : similar operator. **(01)**

d. If $T \in B(H)$ is Hilbert schmidth operator then $\|T\|_2 =$ ______________. **(01)**

e. Is the Following statement is true or false? $B_1(H)$ is an ideal in $B(H)$. **(01)**

f. If $T \in B(H)$ and E Orthonormal basis then$tr(T) =$ ______________. **(01)**

g. Define : trace class operator. **(01)**

Q-5 **Attempt all questions** **(14)**

a) If $T \in B(H)$ is normal , show that$\|T\| = \sup\{| < Tx, x > | : x \in H, \|x\| \leq 1\}$. **(07)**

b) Prove that every normal operator has a non-trivial invariant subspace. **(07)**

OR

Q-5 **Attempt all Questions** **(14)**

a) Suppose $T \in B(H)$ is normal and E is its spectral decomposition. If $f \in C(\sigma(T))$ and if $\omega_0 = f^{-1}(0)$, show that $N\big(f(T)\big) = R(E(\omega_0))$. **(07)**

b) Suppose $T \in B(H)$ is normal and compact. Then show that $f(T)$ is compact if $f \in C(\sigma(T))$
and $f(0) = 0$. Dose the converse hold? **(07)**

Q-6 **Attempt all questions** **(14)**

a) Prove that $tr(ST) = tr(TS)$, $S \in B(H)$ and $T \in B_1(H)$ **(07)**

b) Let $T \in B_1(H)$ and E be an Orthonormal basis then show that $\sum | < Te, e > | < \infty$. **(07)**

OR

Q-6 **Attempt all Questions** **(14)**

a) Let $T \in B_2(H)$ then show that $B_2(H)$ is an ideal in $B(H)$ and $\|\circ\|_2$ is norm on $B_2(H)$. **(07)**

b) If E and F are two Orthonormal basis for H then show that each $T \in B(H)$ **(07)**

$$\sum_{e \in E} \|Te\|^2 = \sum_{f \in F} \|T^* e\|^2 = \sum_{e \in E} \sum_{f \in F} | < Te, f > |^2$$